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A broken gauge approach to gravitational mass and charge

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Abstract: We argue that a spontaneous breakdown of local Weyl invariance offers a mechanism in which gravitational interactions contribute to the generation of particle masses and their electric charge. The theory is formulated in terms of a spacetime geometry whose natural connection has both dynamic torsion and non-metricity. Its structure illuminates the role of dynamic scales used to determine measurable aspects of particle interactions and it predicts an additional neutral vector boson with electroweak properties.

Keywords: Spontaneous Symmetry Breaking, Space-Time Symmetries, Classical Theories of Gravity.

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1. Introduction

In this article we describe a symmetry breaking mechanism for the emergence of gravitational mass and electric charge from interactions that include gravitation. The view is taken that any fundamental theory of masses should involve all the basic interactions on some scale and that an effective theory of gravitation in terms of a small number of relevant fields in four-dimensional spacetime may be sufficient to discern a mechanism of mass generation including gravitation. The theory below will be formulated in terms of a spacetime geometry more general than that adopted by Einstein, Weyl, Cartan and others [1]–[4] since it offers a new approach to the breaking of a Weyl symmetry analogous to the breaking of internal symmetries in the standard model of the fundamental particle interactions.

Since representations of the Weyl group are intimately connected with the assignment of physical units to dimensioned quantities in any theory with Weyl symmetry and the latter are related to experimental predictions, it is useful to first make precise the treatment of physical dimensions in a theory that maintains spacetime diffeomorphism covariance throughout. Dimensional analysis touches on the fundamentals of physical theory [6, 7] and is particularly acute in theories such as gravitation formulated in terms of some spacetime geometry. Tensor operations (such as parallel transport, contraction, integration and differentiation) permit the construction of tensor equations for any classical theory. The interpretation of the theory is in large measure based on the correspondence of scalars with the results of experimental operations. It is also a result of experience that a hierarchy of broken symmetries of various kinds can be used to classify the most relevant scalars in Nature.

Real spacetime geometry offers a framework to formulate particle and gravitational field interactions on a manifold [5]. By real geometry, we mean here the assignment of a metric tensor field g and a (Koszul) linear connection ∇ on spacetime. The former has lorentzian signature while the latter is permitted to have non-zero torsion

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.1}$$

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for any vector fields X, Y and non-metricity $S = \nabla g$. Given an arbitrary local basis of vector fields $\{X_a\}$, the most general linear connection is specified locally by a set of 16 1-forms $\Lambda^a{}_b$ where:

$$\nabla_{X_a} X_b = \Lambda^c{}_b(X_a)\, X_c\,. \tag{1.2}$$

Such a connection can be fixed by specifying on spacetime the (2,0) symmetric metric tensor field g, the (2-antisymmetric,1) tensor field T and the (3,0) tensor field S, symmetric in its last two arguments. It is straightforward to determine the connection in terms of these tensors. Indeed since ∇ is defined to commute with contractions and reduce to differentiation on scalars, it follows from the relation

$$X(g(Y,Z)) = S(X,Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1.3}$$

that

$$\begin{aligned} 2\,g(Z,\nabla_X Y) \;=\; & X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) - \\ & -g(Y,[X,Z]) - g(Z,[Y,X]) - g(X,T(Y,Z)) - g(Y,T(X,Z)) - \\ & -g(Z,T(Y,X)) - -S(X,Y,Z) - S(Y,Z,X) + S(Z,X,Y)\,, \end{aligned} \tag{1.4}$$

where X, Y, Z are any vector fields. The curvature operator $\mathbf{R}_{X,Y}$ of ∇ defined by

$$\mathbf{R}_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.5}$$

is a type-preserving tensor derivation on the algebra of tensor fields. The general (3,1) curvature tensor R of ∇ is defined by

$$R(X,Y,Z,\beta) = \beta(\mathbf{R}_{XY} Z)\,, \tag{1.6}$$

where β is an arbitrary 1-form. This tensor gives rise to a set of local curvature 2-forms $R^a{}_b$:

$$R^a{}_b(X,Y) = \frac{1}{2}\, R(X,Y,X_b,e^a)\,, \tag{1.7}$$

where $\{e^c\}$ is any local basis of 1-forms dual to $\{X_c\}$. In terms of the contraction operator ι_X with respect to X one has $\iota_{X_b}\, e^a \equiv \iota_b\, e^a = e^a(X_b) = \delta^a{}_b$. In terms of the connection forms

$$R^a{}_b = d\,\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b\,. \tag{1.8}$$

The general Ricci tensor is $Ric \equiv R(-,-,X_a,e^a)$ with curvature scalar

$$\mathcal{R} \equiv G(e^a,e^b)\, Ric(X_b,X_a)\,,$$

where $G(e^a,e^b)\, g(X_b,X_c) = \delta^a_c$. In a similar manner the torsion tensor gives rise to a set of local torsion 2-forms T^a:

$$T^a(X,Y) \equiv \frac{1}{2}\, e^a(T(X,Y)) \tag{1.9}$$

which can be expressed in terms of the connection forms as

$$T^a = d\,e^a + \Lambda^a{}_b \wedge e^b\,. \tag{1.10}$$

Measurable quantities $\mathcal{O}[g, \nabla, C, \Phi, \ldots]$ are generally functionals of the geometry, spacetime chains C (such as worldlines, worldtubes, spacelike hypersurfaces etc.), tensor and spinor fields Φ carrying representations of local Lie groups and constant physical parameters. For example local p-forms α generate $\mathcal{O}[\alpha, C_p] = \int_{C_p} \alpha$ for some p-chain C_p.

Let $[\lambda_1, \lambda_2, \lambda_3, \ldots]$ be a list of symbols that can be used to define the physical attributes of a quantity. This list encodes the *dimension* of the quantity. If $[d_1, d_2, d_3, \ldots]$ is a list of (in general fractional) numbers then the (p,q)-type tensor field $T^{(p,q)}_{[d_1, d_2, d_3, \ldots]}$ can be ascribed the arbitrary dimension $\lambda_1^{d_1}\, \lambda_2^{d_2}\, \lambda_3^{d_3}\, \ldots$. Once dimensions are assigned the theory is dimensionally coherent provided only quantities with the same dimension are added together. Tensor fields of different dimension can be multiplied together and:

$$T^{(p_1,q_1)}_{[d_1, d_2, d_3, \ldots]} \otimes T^{(p_2,q_2)}_{[d'_1, d'_2, d'_3, \ldots]} = T^{(p_1+p_2, q_1+q_2)}_{[d_1+d'_1, d_2+d'_2, d_3+d'_3, \ldots]}\,.$$

Scalar fields are special cases in which $\otimes$ is replaced by ordinary scalar multiplication and for a non-zero scalar field one may multiply by its reciprocal. Once a local frame of vector and/or covector fields is choosen for some spacetime region, the components of tensors are scalars obtained by contraction. Although the (p,q)-type of a tensor changes under the action of ∇ and exterior differentiation d, its dimension does not. Abbreviating $[T^{(p,q)}_{[d_1, d_2, d_3, \ldots]}] \equiv [T] = [\lambda_1^{d_1}\, \lambda_2^{d_2}\, \lambda_3^{d_3}\, \ldots]$ for any tensor T then

$$\begin{aligned}
[T(X, Y, \ldots, \alpha, \beta, \ldots)] &= [T]\,[X]\,[Y] \cdots [\alpha]\,[\beta] \cdots \\
[\nabla_X T] &= [X][T] \\
\left[\int_C \alpha\right] &= [\alpha]\,.
\end{aligned}$$

Units are numerical measures assigned to physical dimensions. A local choice of units can be made by choosing a units frame $[\alpha^0_{[1,0,0,\ldots]}, \alpha^0_{[0,1,0,\ldots]}, \alpha^0_{[0,0,1,\ldots]}, \ldots]$ where $\alpha^0_{[0,0,\ldots,1,0,0\ldots]}$ is a non-zero constant scalar field with dimensions λ_k. Such a field will be called a *constant fiducial scalar*. A priori there is no reason to demand that it be dynamically related to any element in the theory. In a theory with a choice of MKS dimensions $[\lambda_1 \equiv Mass,\ \lambda_2 \equiv Length,\ \lambda_3 \equiv Time]$ a local units frame is given by the scalar fields $[kg, m, sec]$ whose constant values at each event are $[1, 1, 1]$. Just as an everywhere non-zero scalar field with dimensions can be used to change the dimensions of any quantity by multiplication, so a general non-zero dimensioned scalar field can be similarly used to change a units frame.

At any event p in spacetime the scalar field quantity $\Phi_{[d_1, d_2, \ldots, d_n]}$ is said to have the numerical *value*

$$\frac{\Phi_{[d_1, d_2, \ldots, d_n]}|_p}{\left(\alpha^0_{[1,0,0,\ldots]}\right)^{d_1} \cdots \left(\alpha^0_{[0,0,\ldots,1]}\right)^{d_n}}$$

in a units frame constructed from constant fiducial scalars. Clearly a constant scalar has a constant value in such a frame. In a units frame constructed from non-constant fiducial scalars the *value* of a quantity may vary from event to event. Thus a meaningful comparison of quantities with the same dimensions is obtained by taking their ratio.

Most experimental measurements are interpreted in some canonical units frame. In relativistic physics one may base a series of measurements on the elapsed time between

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events according to a choice of clock. In this way spatial intervals and particle accelerations can be constructed consistent with special relativity and newtonian gravitation. In Einstein gravitation the physics arises naturally from a (pseudo-)riemannian spacetime geometry based on a Levi-Civita connection associated with a metric tensor g. It is convenient to induce dimensions from the assignment $[g] = L^2$ where L is a constant fiducuial *length* scalar. Taking c to be a constant *speed* scalar, let $T = L/c$ be a constant fiducial *time* scalar. The constant scalars Λ_0 (which we call *action*) and Q_e (which we call *electric charge*) are useful in the development of a theory with charged masses. An MKS units frame $[m, sec, Joule-sec, Coulombs]$ with values $[1, 1, \hbar = 1.0545 \times 10^{-24}, e_0 = 1.6021 \times 10^{-19}]$ is traditional for the theory. Whether such fiducial frames exist more generally, either locally or globally on spacetime is a question for experiment and can only be answered within the framework of physical laws.

It is clear that until an assignment of units is made only dimensionless quantities are meaningful in a physical theory. It has been suggested that the values of certain constant fiducial scalars (e.g $\hbar, e_0$) arise from a particular choice of a spacetime metric solution to the equations for gravitation and that in the presence of dynamical scalar fields new units frames can be choosen in which some of the so called *constants of nature* (such as the newtonian gravitational coupling constant G) have values that vary with cosmological time [8, 9, 10]. Einstein's metric theory of gravity offers an unambiguous way to define time intervals and if c, e_0 and $\hbar$ are constant scalars in such a description a complete edifice can be constructed in terms of a single constant fundamental length scalar. Masses $\mu = \hbar/c\lambda_0$ are given naturally in terms of their Compton wavelengths λ_0 and G appears to be constant in this framework. If the theory is generalised to include dynamical scalars and a more general spacetime geometry then the role of the fiducial scalars is more diffuse. Since a dynamic theory of spacetime geometry of necessity has implications for its own experimental interpretation the latter may depend on which scale or epoch of the universe the theory purports to describe.

We take the view that, in the current epoch of the universe, the interpretation of the gravitational mass of *particles* associated with fields in a generalised theory be based on its reduction to equations that involve the Einstein tensor $Ein = Ric - \frac{1}{2}\mathcal{R}\, g$ of the Levi-Civita $(T = 0,\ S = 0)$ connection associated with the spacetime metric g satisfying :

$$Ein = \mathcal{T} \tag{1.11}$$

for some source tensor $\mathcal{T}$. With $[g] = L^2$, then $[Ein] = [1]$ and the equation $\mathcal{T} = 8\pi G\mathbf{T}$ identifies $\mathbf{T}$ as the einsteinian mass-energy stress tensor. For any spacetime observer Z with $g(Z, Z) = -c^2$ the scalar $\rho = \mathbf{T}(Z, Z) > 0$ is a physical mass density. If the above equation admits a static solution with

$$g = -c^2\left(1 - \frac{2\varphi}{c^2} + \cdots\right) d\,t \otimes d\,t + \left(1 + \frac{2\varphi}{c^2} + \cdots\right)\hat{g} \tag{1.12}$$

in local coordinates (t, x^i) and with $\hat{g} = \sum_{i=1}^{3} d\,x^i \otimes d\,x^i$, then in the non-relativistic weak field limit, φ may be identified with the newtonian potential satisfying

$$(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3})\varphi = 4\pi G\,\rho\,. \tag{1.13}$$

The Levi-Civita geodesic motion of a neutral test particle then reduces to the newtonian equation of motion:

$$\frac{d\,X_i(t)}{d\,t^2} = -\partial_{x_i}\varphi(X_1(t), X_2(t), X_3(t))\,. \tag{1.14}$$

In a regular three dimensional spacelike domain $\mathcal{B}$ the gravitational mass M_G associated with φ is given as

$$M_G = \frac{1}{4\pi G}\int_{\partial\mathcal{B}} \hat{\star}\, d\varphi \tag{1.15}$$

in terms of the Hodge map of $\hat{g}$.

In a theory with Weyl scaling it is also important to be able to identify electric charge. Classical electromagnetism is associated with a closed 2-form $\mathcal{F}$ and a closed 3-form j on spacetime:

$$d\,\mathcal{F} = 0 \tag{1.16}$$

$$d\,j = 0\,. \tag{1.17}$$

In a (rationalised) MKS units frame, $\mathcal{F}$ is assigned dimensions $[Q_e/\epsilon_0]$ where the fiducial scalar ϵ_0 is assigned dimensions $[\frac{{Q_e}^2T^2}{ML^3}]$. The conserved current associated with the Maxwell 2-form $\mathcal{F}$ is then defined by the field equation

$$j = d * (\epsilon_0\mathcal{F}) \tag{1.18}$$

so $[j] = [Q_e]$. The electric 1-form field E and electric charge density ρ_e defined by the observer Z are $E = \frac{1}{c}\iota_Z\mathcal{F}$ and $\rho_e = \frac{1}{c}\iota_Z(*j)$, respectively. Since $\mathcal{F}$ is closed there exists a potential 1-form $\mathcal{A}$ on a regular domain of spacetime such that $\mathcal{F} = d\,\mathcal{A}$. If there exists a static field configuration with $E = d\,\varphi_e$ then the electrostatic potential φ_e satisfies

$$\left(\partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}\right)\varphi_e = \frac{\rho_e}{\epsilon_0} \tag{1.19}$$

in the above metric in the weak gravitational field limit. The electric charge $Q_e[\varphi_e, \mathcal{B}]$ in Coulombs associated with φ_e in a regular spacelike three dimensional domain $\mathcal{B}$ is given as

$$Q_e[\varphi_e, \mathcal{B}] = \epsilon_0\int_{\partial\mathcal{B}} \hat{\star}\, d\varphi_e \tag{1.20}$$

in terms of the Hodge map of $\hat{g}$.

The fundamental difference between *mass* and *electric charge* arises from the structure of the sources $\mathcal{T}$ and j. In the absence of local Weyl symmetry the structure of j depends on other fields Φ carrying representations of some U(1) group. Thus if θ is an arbitrary real scalar field on spacetime and e some *charge parameter* with $[e\,\theta] = 1$,

$$\Phi \mapsto \exp(-ie\theta)\,\Phi \tag{1.21}$$

under the U(1) group action. If the above field equations (1.16), (1.18) are generated from a suitable locally U(1) invariant action in which the 1-form $\mathcal{A}$ carries a representation of a U(1) connection then charge conservation is guaranteed and the field equations are also U(1) invariant. In a locally Weyl covariant theory it is more natural to work with a dimensionless connection A and recover the Maxwell potential $\mathcal{A}$, Maxwell field $\mathcal{F}$ and MKS electric charge sources by transforming with a fiducial scalar having dimensions $[Q_e/\epsilon_0]$.

To implement a theory of dynamic mass generation for the current epoch we seek a mechanism to break a theory with local Weyl symmetry. To this end we must first construct a locally Weyl covariant theory of gravitation where certain field elements Φ also carry representations of the scaling group. Thus if σ is an arbitrary real scalar field on spacetime

$$\Phi \mapsto \exp(-q\sigma)\,\Phi \tag{1.22}$$

under the Weyl group action. The dimensionless real constant q is called the *Weyl or scale charge parameter* of the representation. Scale charges are relative to the representation carried by a class of metric tensors $[g]$, elements of which are equivalent under

$$g \mapsto \exp(2\sigma)\,g\,. \tag{1.23}$$

We assign zero dimensions to σ so that elements on the same Weyl group orbit all have the same dimension. A Weyl connection can be represented by a dimensionless 1-form Q such that

$$Q \mapsto Q + d\,\sigma \tag{1.24}$$

under a local Weyl transformation. In terms of Q the exterior Weyl covariant derivative of a generic p-form Φ^p_q with Weyl charge q is defined to be:

$$\mathcal{D}\Phi^p_q = D\Phi^p_q + q\,Q\,\wedge\Phi^p_q\,, \tag{1.25}$$

where D is the exterior covariant derivative that maintains the dimension, scale charge and covariance of $\mathcal{D}\Phi^p_q$ under all local group transformations in the theory. Thus under the Weyl group $\mathcal{D}\Phi^p_q \mapsto \exp(-q\sigma)\,\mathcal{D}\Phi^p_q$. For pure Weyl scalars $D = d$.

A class of Hodge maps $[\star]$ is associated with $[g]$. One may readily verify from (1.25) that

$$\mathcal{D}\star\Phi^p_q = D\star\Phi^p_q + (q-(4-2p))\,Q\,\wedge\star\Phi^p_q\,. \tag{1.26}$$

If one denotes Φ^p_q by $\left\{{p \atop q}\right\}\Phi$ then it is also straightforward to verify the following rules:

$$\begin{aligned}
\star\left(\left\{{p \atop q}\right\}\Phi\right) &= \left\{{4-p \atop q-(4-2p)}\right\}(\star\,\Phi)\\
\mathcal{D}\left(\left\{{p \atop q}\right\}\Phi\right) &= \left\{{p+1 \atop q}\right\}(\mathcal{D}\,\Phi)\\
\left(\left\{{p_1 \atop q_1}\right\}\,\Phi_1\right)\wedge\left(\left\{{p_2 \atop q_2}\right\}\,\Phi_2\right) &= \left\{{p_1+p_2 \atop q_1+q_2}\right\}\,(\Phi_1\wedge\Phi_2)\,.
\end{aligned}$$

In order to construct a locally Weyl invariant action principle for our models we take the dynamic Weyl connection 1-form Q proportional to the metric trace of the non-metricity tensor $S = \nabla g$ in a geometry with:

$$S = 2\epsilon\;Q\otimes g\,, \tag{1.27}$$

where $\epsilon^2 = 1$. In order to induce the appropriate behaviour of Q under a change of Weyl gauge, we adopt the Weyl transformation rule

$$\nabla \mapsto \nabla \tag{1.28}$$

for the connection. From (1.23) this induces the transformation:

$$Q \mapsto Q + \epsilon\, d\sigma \tag{1.29}$$

so we take $\epsilon = 1$. Since ∇ preserves Weyl covariance the torsion tensor given by (1.1) remains invariant as does the curvature operator $\mathbf{R}_{XY}$.[1] In a field of local *orthonormal* co-frames $\{e^a\}$ with $g = \eta_{a\,b} e^a \otimes e^b$, one has under a Weyl scaling

$$e^a \mapsto \exp(\sigma)\, e^a\,. \tag{1.30}$$

Thus the torsion 1-forms T^a transform as

$$T^a \mapsto \exp(\sigma)\, T^a\,. \tag{1.31}$$

2. Weyl invariant actions

To facilitate the derivation of the field equations from an action it is convenient to adopt a class of local g-orthonormal co-frames $\{e^a\}$ and their duals $\{X_a\}$. In the absence of internal gauge symmetries D is then determined by the connection 1-forms $\{\Lambda^a{}_b\}$ in such bases. Since $\eta_{a\,b} = g(X_a, X_b)$ and $(D\eta_{a\,b})(X) = (\nabla_X g)(X_a, X_b)$ for all X:

$$D\eta_{ab} = d\eta_{ab} - \Lambda^c{}_a \eta_{cb} - \Lambda^c{}_b \eta_{ac} = -(\Lambda_{ab} + \Lambda_{ba}) = -2Q_{ab}\,, \tag{2.1}$$

in terms of the non-metricity 1-forms Q_{ab}. Thus in a geometry with $\nabla_X g = 2\,Q(X)\,g$ it follows that

$$Q_{ab} = -Q\,\eta_{ab}\,. \tag{2.2}$$

A geometry with this S (but zero torsion) was first suggested by Weyl [1, 2] and provided the precursor to the modern gauge approach to particle interactions. Under the local Weyl scalings above:

$$Q_{ab} \mapsto Q_{ab} - \eta_{ab}\, d\sigma \tag{2.3}$$

and it follows from the definition of the connection forms (in any basis) that $[\Lambda^a{}_b] = 1$, $[Q] = 1$, $[Q_{ab}] = 1$. The connection 1-forms in any orthonormal basis can be decomposed into their anti-symmetric Ω_{ab} and symmetric parts Q_{ab},

$$\Lambda_{ab} = \Omega_{ab} + Q_{ab}\,. \tag{2.4}$$

The anti-symmetric part can be further decomposed in a unique way according to

$$\Omega_{ab} = \omega_{ab} + K_{ab} + q_{ab}\,. \tag{2.5}$$

[1] The more general rule $\nabla \mapsto \nabla + d\,f \otimes$ acting on arbitrary tensor fields is possible and also preserves $\mathbf{R}_{XY}$ but serves only to complicate the Weyl transformation properties of the torsion tensor [11].

Here the (torsion-free) Levi-Civita connection 1-forms $\omega^a{}_b$ satisfy the structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0\,. \tag{2.6}$$

The contortion 1-forms $K^a{}_b$ fix the torsion:

$$K^a{}_b \wedge e^b = T^a\,, \tag{2.7}$$

and

$$q_{ab} = -\iota_{X_a} Q_{bc} e^c + \iota_{X_b} Q_{ac} e^c\,. \tag{2.8}$$

In the presence of torsion and non-metricity the exterior covariant derivative D above satisfies

$$\begin{aligned} D * e_a &= *(e_a \wedge e_b) \wedge T^b + 4Q \wedge *e_a\,, \\ D * (e_a \wedge e_b) &= *(e_a \wedge e_b \wedge e_c) \wedge T^c + 4Q \wedge *(e_a \wedge e_b)\,, \\ D * (e_a \wedge e_b \wedge e_c) &= *(e_a \wedge e_b \wedge e_c \wedge e_d) T^d + 4Q \wedge *(e_a \wedge e_b \wedge e_c)\,, \\ D * (e_a \wedge e_b \wedge e_c \wedge e_d) &= 4Q * (e_a \wedge e_b \wedge e_c \wedge e_d) \end{aligned} \tag{2.9}$$

in terms of the Hodge map with $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = *1$. These relations prove of value in the following sections.

An action $\mathcal{S} = \int_M \mathbf{\Lambda}$ is locally Weyl invariant for some 4-form $\mathbf{\Lambda}$ if, for some 3-form ψ (with compact support on spacetime M), the transformation

$$\mathbf{\Lambda} \mapsto \mathbf{\Lambda} + d\psi$$

is induced by Weyl scalings. We begin by considering a Weyl invariant action $\mathcal{S}[g, \nabla, \alpha]$ constructed with the aid of a real dynamic scalar field having $[\alpha] = L^{-1}$ and Weyl charge 1, $\alpha \mapsto \exp(-\sigma)\,\alpha$. Since metric variations are induced by orthonormal co-frame variations we write:

$$\mathcal{S}[\{e^a\}, \{\Lambda^a{}_b\}, \alpha] = \int_M \mathbf{\Lambda}_1 \tag{2.10}$$

with

$$\mathbf{\Lambda}_1 = \mathbf{\Lambda}_0 \left(\alpha^2 R^a{}_b \wedge *(e_a \wedge e^b) - \frac{\gamma}{2} \mathcal{D}\alpha \wedge *\mathcal{D}\alpha - \frac{1}{2} dQ \wedge *dQ \right).$$

The coupling γ is a real dimensionless constant while $\mathbf{\Lambda}_0$ is a constant with the dimensions of action. Co-frame variations of (2.10) yield the gravitational field equations

$$\alpha^2 G_a = \gamma \tau_a[\alpha] + \hat{\tau}_a[dQ]\,, \tag{2.11}$$

where the generalised Einstein 3-forms

$$G_a \equiv -R^b{}_c \wedge *(e_a \wedge e_b \wedge e^c) \tag{2.12}$$

and the source 3-forms are given by

$$\tau_a[\alpha] = \frac{1}{2} (\iota_{X_a} \mathcal{D}\alpha * \mathcal{D}\alpha + \mathcal{D}\alpha \wedge \iota_{X_a} * \mathcal{D}\alpha)\,, \tag{2.13}$$

and

$$\hat{\tau}_a[dQ] = \frac{1}{2}(\iota_{X_a} dQ \wedge *dQ - dQ \wedge \iota_{X_a} * dQ)\,. \tag{2.14}$$

The α field variation of (2.10) yields

$$2\alpha R^a{}_b \wedge *(e_a \wedge e^b) + \gamma \mathcal{D} * \mathcal{D}\alpha = 0\,. \tag{2.15}$$

The trace of (2.11) follows by left exterior multiplication by e^a. By comparing with (2.15) the terms proportional to the scalar curvature can be eliminated yielding:

$$\frac{\gamma}{2}\mathcal{D} * \mathcal{D}\alpha^2 = 0\,. \tag{2.16}$$

Connection variations of (2.10) are carried out under the constraint (2.2), a condition that may be imposed by the method of Lagrange multipliers. Such variations need care since one cannot raise and lower indices freely under the covariant derivatives. Noting that the variation $\delta Q = -\frac{1}{4}\delta\Lambda^a{}_b \eta^b{}_a$ one obtains

$$2D(\alpha^2 * (e_a \wedge e^b)) - \eta^b{}_a(\gamma\alpha * \mathcal{D}\alpha + d * dQ) = 0\,. \tag{2.17}$$

The symmetric part of (2.17) gives the Q-field equation

$$d * dQ + \gamma\alpha * \mathcal{D}\alpha = 0\,. \tag{2.18}$$

Note that no further equations arise from exterior differentiation of (2.18) since (2.16) is recovered. On the other hand, by lowering an index in (2.17), picking up a new term proportional to the non-metricity in the process, and setting the anti-symmetric part of the resulting equation to zero one finds:

$$d\alpha^2 \wedge *(e_a \wedge e_b) + \alpha^2 * (e_a \wedge e_b \wedge e_c) \wedge T^c + 2\alpha^2 Q \wedge *(e_a \wedge e_b) = 0\,. \tag{2.19}$$

This is an algebraic equation that can be solved uniquely for the torsion 2-forms:

$$T^a = e^a \wedge \frac{d\alpha}{\alpha} + e^a \wedge Q\,.$$

The corresponding contortion 1-forms are

$$K_{ab} = -e_b\, \iota_{X_a}\left(\frac{d\alpha}{\alpha} + Q\right) + e_a\, \iota_{X_b}\left(\frac{d\alpha}{\alpha} + Q\right). \tag{2.20}$$

Therefore the connection 1-forms become

$$\Lambda_{ab} = \omega_{ab} - e_b\,\frac{\iota_{X_a} d\alpha}{\alpha} + e_a\,\frac{\iota_{X_b} d\alpha}{\alpha} - \eta_{ab} Q\,. \tag{2.21}$$

The curvature 2-forms $R^a{}_b$ of this connection may be written in terms of the curvature 2-forms $R(\omega)^a{}_b$ of the Levi-Civita connection as follows:

$$\begin{aligned} R^a{}_b = {} & R(\omega)^a{}_b - D(\omega)\left(\frac{\iota^a d\alpha}{\alpha}\right) \wedge e_b + D(\omega)\left(\frac{\iota_b d\alpha}{\alpha}\right) \wedge e^a + \\ & + \frac{\iota^a d\alpha}{\alpha}\frac{d\alpha}{\alpha} \wedge e_b - \frac{\iota_b d\alpha}{\alpha}\frac{d\alpha}{\alpha} \wedge e^a + \\ & + * \left(\frac{d\alpha}{\alpha} \wedge *\frac{d\alpha}{\alpha}\right) e^a \wedge e_b - \eta^a{}_b dQ\,. \end{aligned} \tag{2.22}$$

The *(torsion-free) Weyl connection 1-forms* [1, 3, 12, 13] are determined by the difference

$$\Gamma^{a}{}_{b} \equiv \Lambda^{a}{}_{b} - K^{a}{}_{b}\,. \tag{2.23}$$

Thus

$$\Gamma_{ab} = \omega_{ab} + e_b\, \iota_{X_a} Q - e_a\, \iota_{X_b} Q - \eta_{ab} Q\,. \tag{2.24}$$

The same curvature 2-forms $R^{a}{}_{b}$ above may also be expressed in terms of the curvature 2-forms $R(\Gamma)^{a}{}_{b}$ of this Weyl connection:

$$\begin{aligned} R^{a}{}_{b} = {} & R\,(\Gamma)^{a}{}_{b} - D\,(\Gamma)\left(\iota^a\left(\frac{d\alpha}{\alpha} + Q\right)\right) \wedge e_b + D\,(\Gamma)\left(\iota_b\left(\frac{d\alpha}{\alpha} + Q\right)\right) \wedge e^a - \\ & - \left(\iota^a\left(\frac{d\alpha}{\alpha} + Q\right)\right) - \left(\frac{d\alpha}{\alpha} + Q\right) \wedge e_b \left(\iota_b - \left(\frac{d\alpha}{\alpha} + Q\right)\right) - \left(\frac{d\alpha}{\alpha} + Q\right) \wedge e^a + \\ & + * \left(\left(\frac{d\alpha}{\alpha} + Q\right) \wedge * \left(\frac{d\alpha}{\alpha} + Q\right)\right) e^a \wedge e_b - 2\left(\iota^a\left(\frac{d\alpha}{\alpha} + Q\right)\right) Q \wedge e_b\,. \end{aligned} \tag{2.25}$$

The above field equations (2.11), (2.16), (2.18) constitute a Weyl covariant theory of gravity in which solutions with a varying everywhere non-zero α can be used to determine a dynamic units frame. A local Weyl gauge can be found that transforms any such solution for α to a constant α_0. In such a gauge the equations above take the form:

$$\begin{aligned} {\alpha_0}^2 G_a &= \gamma {\alpha_0}^2 \tau_a[Q] + \hat{\tau}_a[dQ]\,, \\ d * dQ + \gamma {\alpha_0}^2 * Q &= 0\,, \\ d * Q &= 0 \end{aligned} \tag{2.26}$$

where

$$\tau_a[Q] = \frac{1}{2}(\iota_{X_a} Q * Q + Q \wedge \iota_{X_a} * Q)\,. \tag{2.27}$$

It should be stressed that choosing such a gauge is in no way equivalent to breaking local Weyl symmetry. It follows from (2.22) that this is an Einstein-Proca system and any solution to this system will generate a class of Weyl gauge equivalent solutions. Since only the light-cone conformal spacetime structure is a Weyl class invariant such a theory has no preferred mass scale.

One way to break the Weyl covariance of the above action is to consider the theory with $Q = 0$ and $\gamma \neq 0$. In this case it follows from (2.22) that the gravitational field equations can be written in terms of the Einstein tensor of the Levi-Civita connection. One then recovers the Brans-Dicke theory (with the Brans-Dicke scalar α^2) in the absence of matter [14, 15, 16]. The Brans-Dicke coupling parameter ω is identified from $\gamma = 2\omega + 3$. The effect of the contribution by the scalar function α in (2.22) can be identified with the so-called improved stress tensor [17, 18]. We shall proceed differently and maintain the presence of the geometrical field Q.

It is also instructive to express the action (2.10) in terms of the Weyl curvature scalar $R^{a}{}_{b}(\Gamma) \wedge *(e_a \wedge e^b)$:

$$\frac{\mathbf{\Lambda}_1}{\mathbf{\Lambda}_0} = \frac{\alpha^2}{2} R^{a}{}_{b}(\Gamma) \wedge *(e_a \wedge e^b) - \frac{\gamma - 6}{2} \mathcal{D}\alpha \wedge * \mathcal{D}\alpha - \frac{1}{2} dQ \wedge * dQ + \quad (\mathrm{mod}\ d)\,. \tag{2.28}$$

With $\gamma = 0$, this action was considered by Dirac in ref. [9] with an additional term proportional to α^4 (considered in the next section). It is also possible to express the action in terms of the Levi-Civita curvature scalar $R^a{}_b(\omega) \wedge *(e_a \wedge e^b)$:

$$\frac{\mathbf{\Lambda}_1}{\mathbf{\Lambda}_0} = \frac{\alpha^2}{2} R^a{}_b(\omega) \wedge *(e_a \wedge e^b) - \frac{\gamma - 6}{2} d\alpha \wedge *d\alpha - \frac{\gamma}{2} d\alpha^2 \wedge *Q - \\ - \frac{1}{2} dQ \wedge *dQ - \frac{\gamma}{2} \alpha^2 Q \wedge *Q + \quad (\text{mod } d)\,. \tag{2.29}$$

With $Q = 0$ and $\gamma = 0$ this action was considered by Anderson in ref. [19] as the scale invariant limit of the Brans-Dicke theory.

3. Mass generation

A theory with no explicit scale does not describe the world in its current epoch. Particles are observed as field quanta with definite masses and electric charges and the classical world is distinguished from quantum phenomena by actions that are large compared with the Planck unit of action. Inspired by the abelian Higgs model in electrodynamics we now enlarge the theory to include a local U(1) symmetry group. Thus a real U(1) gauge connection 1-form A is introduced along with a complex scalar field β transforming as

$$\beta \mapsto \exp(-ie\theta)\beta \tag{3.1}$$

under

$$A \mapsto A + d\theta\,, \tag{3.2}$$

where e is a dimensionless U(1) charge parameter. Naturally $[e\,\theta] = 1$ and the 1-form A will be taken as inert under Weyl scalings with $[A] = 1$ while β has constant dimensionless Weyl charge q:

$$\beta \mapsto \exp(-q\sigma)\beta\,. \tag{3.3}$$

Thus to maintain both these local symmetries the full gauge covariant exterior derivative of β becomes:

$$\mathcal{D}\beta \equiv d\beta + ieA\beta + qQ\beta \tag{3.4}$$

while its U(1) gauge covariant exterior derivative will be written

$$D_A\beta = d\beta + ieA\beta\,.$$

With $\beta^\dagger \mapsto \exp(ie\theta)\beta^\dagger$ under U(1) and

$$\mathcal{D}\beta^\dagger \equiv d\beta^\dagger - ieA\beta + qQ\beta^\dagger\,,$$

an additional Weyl and U(1) invariant contribution to the previous action with

$$\mathbf{\Lambda}_2 = \mathbf{\Lambda}_0 \left(-\frac{1}{2}\alpha^{2-2q} \mathcal{D}\beta^\dagger \wedge *\mathcal{D}\beta - \frac{1}{2}\, dA \wedge *dA - V(|\beta|, \alpha) * 1 \right) \tag{3.5}$$

is now considered. The Weyl charge assignment for β dictates $[\beta] = [\beta^\dagger] = L^{-q}$. The Weyl and U(1) invariant interaction potential V, depending on three dimensionless constants

c_1, c_2, λ_3, will be responsible for the *simultaneous spontaneous breakdown* of the U(1) and Weyl symmetries:

$$V(|\beta|, \alpha) = c_1 \alpha^{4-4q} |\beta|^4 - c_2\ \alpha^{4-2q} |\beta|^2 + \lambda_3 \alpha^4 . \tag{3.6}$$

One may envisage that the magnitudes and signs of the constants in V control the breakdown as a function of epoch. In the context of cosmology their values might be determined by "matter" or "radiation" temperature. Thus we expect stationary field configurations to occur for different minima of such a potential. By linearising the theory about such solutions we effectively break the above symmetries and seek a mass spectrum for the scalar and gauge field excitations in such a background. The role of the α field in maintaining the Weyl invariance of $V * 1$ in (3.5) should be noted in this context.

The field equations follow from varying the action

$$\mathcal{S}[\{e^a\}, \{\Lambda^a{}_b\}, \alpha, \beta, A] = \int_M (\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2) \tag{3.7}$$

under the same connection constraint as before. The gravitational field equations are

$$\alpha^2 G_a = \gamma \tau_a[\alpha] + \hat{\tau}_a[dQ] + \alpha^{2-2q} \tau_a[\beta] + \hat{\tau}_a[dA] - V * e_a , \tag{3.8}$$

where now

$$\tau_a[\beta] = \frac{1}{4}(\iota_{X_a} \mathcal{D}\beta^\dagger * \mathcal{D}\beta + \mathcal{D}\beta^\dagger \wedge \iota_{X_a} * \mathcal{D}\beta + \iota_{X_a} \mathcal{D}\beta * \mathcal{D}\beta^\dagger + \mathcal{D}\beta \wedge \iota_{X_a} * \mathcal{D}\beta^\dagger) .$$

The α-field equation is

$$\frac{\gamma}{2} \mathcal{D} * \mathcal{D}\alpha^2 + q\alpha^{2-2q} (\mathcal{D}\beta)^\dagger \wedge * \mathcal{D}\beta + \left(4V - \alpha \frac{\partial V}{\partial \alpha}\right) * 1 = 0 . \tag{3.9}$$

The connection field equations include

$$d * dQ + \gamma \alpha * \mathcal{D}\alpha + \frac{q}{2} \alpha^{2-2q} * \mathcal{D}(|\beta|^2) = 0 \tag{3.10}$$

together with the torsion 2-forms

$$T^a = e^a \wedge \frac{d\alpha}{\alpha} + e^a \wedge Q .$$

The β field equation obtained by varying $\beta^\dagger$ is:

$$\frac{1}{2} \mathcal{D}(\alpha^{2-2q} * \mathcal{D}\beta) = \frac{\partial V}{\partial \beta^\dagger} * 1 \tag{3.11}$$

and the A field equation is

$$d * dA + i\frac{e}{2} \alpha^{2-2q} * \Big((D_A\beta)^\dagger \beta - \beta^\dagger (D_A \beta)\Big) = 0 . \tag{3.12}$$

If one writes

$$\beta = |\beta| e^{-ie\phi} \tag{3.13}$$

with $[e\phi] = 1$ so that under local scalings $|\beta| \mapsto e^{-q\sigma}|\beta|, \phi \mapsto \phi$, then under a local U(1) phase change $|\beta| \mapsto |\beta|, \phi \mapsto \phi + \theta$. Note that the combination $B \equiv A - d\phi$ appearing in all field equations is invariant under both scalings and phase transformations and the A-field equation now reads

$$d * dB + e^2 \alpha^{2-2q} |\beta|^2 * B = 0 \,, \tag{3.14}$$

while the β field equation reduces to

$$\mathcal{D}(\alpha^{2-2q} * \mathcal{D}|\beta|) - e^2 |\beta| \alpha^{2-2q} B \wedge *B = \alpha^{4-4q} |\beta| \left(2c_1 |\beta|^2 - c_2 \alpha^{2q} \right) * 1 \,. \tag{3.15}$$

We are interested in an epoch where $c_1 > 0, c_2 > 0$. A stationary ground state solution arises with $\mathcal{D}\alpha = 0$ and $\mathcal{D}\beta = 0$. Since $\mathcal{D}^2\alpha = dQ\alpha$, these solutions for $\alpha \neq 0$ will satisfy $dQ = 0$. Thus we represent this solution in a gauge with $\alpha = \alpha_0$, $Q = 0$, $B = 0$ and $|\beta| = |\beta_0|$ (with the phase of β arbitrary) such that

$$|\beta_0|^2 = \frac{c_2}{2c_1} {\alpha_0}^{2q} \tag{3.16}$$

with

$$\lambda_3 = \frac{{c_2}^2}{4c_1} \,.$$

The latter choice selects a Minkowski metric solution $g = \eta$ corresponding to a Levi-Civita flat ground state solution.[2] Thus the effective potential takes the form

$$V(|\beta|, \alpha) = c_1 \alpha^4 \left(\alpha^{-2q} |\beta|^2 - \frac{c_2}{2c_1} \right)^2 \,. \tag{3.17}$$

To determine the mass spectrum in the broken symmetry phase the field equations must be appropriately linearised about the ground state solution. This is achieved by writing

$$\begin{aligned} \alpha &= \alpha_0 + \epsilon\hat{\alpha} \,, \qquad Q = \epsilon \hat{Q} \,, \qquad g = \eta + \epsilon^2 \hat{g} \,, \\ B &= \epsilon \hat{B} \,, \\ \beta &= (|\beta_0| - \lambda \epsilon \hat{\alpha}) \exp(-ie\phi) \,, \end{aligned} \tag{3.18}$$

where $\lambda = (\gamma/q) {\alpha_0}^{q-1} \sqrt{(2c_1)/(c_2)}$.

Then to order ϵ the wave equation for $\hat{Q}$ is

$$*d*d\,\hat{Q} + \left(\gamma\, {\alpha_0}^2 + \frac{q^2 {\alpha_0}^2 c_2}{2c_1} \right) \hat{Q} = 0 \,. \tag{3.19}$$

This implies

$$d * \hat{Q} = 0 \,. \tag{3.20}$$

Eq. (3.19) admits propagating modes with angular frequency ω and wave number k satisfying the dispersion relation:

$$\omega^2 = c^2 k^2 + c^2 \gamma\, {a_0}^2 + \frac{{a_0}^2 c^2 q^2 c_2}{2c_1} \,. \tag{3.21}$$

[2]By modifying V one might consider symmetry breaking about a de Sitter background.

Similarly to order ϵ the wave equation for $\hat{B}$ is

$$*d*d\hat{B} = -\frac{e^2{\alpha_0}^2 c_2}{2c_1}\hat{B} \tag{3.22}$$

with dispersion relation

$$\omega^2 = c^2k^2 + \frac{e^2c^2{\alpha_0}^2c_2}{2c_1}\,. \tag{3.23}$$

Thus in a units frame with constant $\hbar$ one predicts for $\gamma > 0$, vector particles having positive gravitational masses given by:

$$\begin{aligned} M_{\hat{Q}}^2 &= \frac{\alpha_0^2\hbar^2}{c^2}\left(\gamma + \frac{c_2q^2}{2c_1}\right), \\ M_{\hat{B}}^2 &= \frac{\hbar^2e^2\alpha_0^2c_2}{2c^2c_1}\,. \end{aligned} \tag{3.24}$$

Furthermore using (3.19) equations (3.9), (3.15) for α and β both reduce to order ϵ to

$$*d*d\,\hat{\alpha} + (4c_2{\alpha_0}^2 + \frac{2\alpha_0^2q^2c_2^2}{c_1\gamma})\,\hat{\alpha} = 0\,. \tag{3.25}$$

with dispersion relation

$$\omega^2 = c^2k^2 + 2\,\frac{{\alpha_0}^2{c_2}^2c^2q^2}{\gamma\,c_1} + 4\,{\alpha_0}^2c_2c^2\,. \tag{3.26}$$

Thus the spontaneous breakdown induces a massive excitation with

$$M_{\hat{\alpha}}^2 = \frac{4c_2\alpha_0^2\hbar^2}{c^2}\left(1 + \frac{c_2q^2}{2\gamma c_1}\right) \tag{3.27}$$

for the $\hat{\alpha}$ field. We note that the masses in the broken phase are determined by the parameters $\alpha_0, \gamma, c_1, c_2, e, q$. The scale of the broken theory can be established in terms of any one of these masses.

4. Mass and charge generation

In the previous section the U(1) symmetry was broken in the process of mass generation. Consequently one cannot identify electric charge in the broken phase. The "standard model" however provides a symmetry breaking mechanism that leaves a U(1) symmetry intact compatible with the observation of electric charge conservation. It is of interest to embed this mechanism into a theory with local Weyl symmetry. We restrict to the bosonic sector of the $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$ electroweak theory which contributes an action 4-form

$$\frac{\mathbf{\Lambda}_3}{\mathbf{\Lambda}_0} = -\frac{1}{2}dA\wedge *dA - \frac{1}{2}\mathrm{Tr}(\mathbf{F}\wedge *\mathbf{F}) - \frac{\alpha^{2-2q}}{2}(\mathcal{D}\mathbf{\Phi})^\dagger \wedge *(\mathcal{D}\mathbf{\Phi}) - V(|\mathbf{\Phi}|, \alpha) *1\,, \tag{4.1}$$

where iA is the hypercharge potential 1-form, $\mathbf{F} = d\mathbf{A} + [\mathbf{A}, \mathbf{A}]$ with $\mathbf{A}$ being the SU(2) Lie algebra (with basis $\mathbf{T}_j$) valued potential 1-form. Here the Higgs scalar $\mathbf{\Phi}$ is a complex isodoublet

$$\mathbf{\Phi} = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \tag{4.2}$$

carrying a Weyl representation with Weyl charge q

$$\mathbf{\Phi} \mapsto e^{-q\sigma}\mathbf{\Phi}\,, \tag{4.3}$$

which under a $\mathrm{SU}(2) \times \mathrm{U}(1)$ transformation transforms as

$$\mathbf{\Phi} \mapsto e^{-g\mathbf{\Theta}\cdot\mathbf{T}-\frac{i}{2}g'\theta}\mathbf{\Phi}\,. \tag{4.4}$$

Thus its gauge covariant exterior derivative

$$\mathcal{D}\mathbf{\Phi} = d\mathbf{\Phi} + g\mathbf{A}\mathbf{\Phi} + i\frac{g'}{2}A\mathbf{\Phi} + qQ\mathbf{\Phi}\,, \tag{4.5}$$

where

$$\mathbf{A} = A_j\frac{\mathbf{t}_j}{2i}$$

with Pauli matrices $\{\mathbf{t}_j\}$. To maintain Weyl invariance the potential is constructed as

$$V(|\mathbf{\Phi}|,\alpha) = \lambda\alpha^4(\alpha^{-2q}|\mathbf{\Phi}|^2 - v^2)^2\,, \tag{4.6}$$

where λ and v are real constants. The variational field equations are found from the combined action

$$\mathcal{S}[\{e^a\},\{\Lambda^a{}_b\},\alpha,\mathbf{\Phi},\mathbf{A},A] = \int_M(\mathbf{\Lambda}_1 + \mathbf{\Lambda}_3)\,. \tag{4.7}$$

Coframe variations yield the gravitational field equations

$$\alpha^2 G_a = \gamma\tau_a[\alpha] + \hat{\tau}_a[dQ] + \tau_a[\mathbf{\Phi}] + \hat{\tau}_a[dA] + \hat{\tau}_a[\mathbf{F}] - V(|\mathbf{\Phi}|,\alpha) * e_a\,, \tag{4.8}$$

while α variations give:

$$\frac{\gamma}{2}\mathcal{D} * \mathcal{D}\alpha^2 + q\alpha^{2-2q}(\mathcal{D}\mathbf{\Phi})^\dagger \wedge *\mathcal{D}\mathbf{\Phi} + \left(4V - \alpha\frac{\partial V}{\partial\alpha}\right) * 1 = 0\,. \tag{4.9}$$

The connection variational equation

$$D(\alpha^2 * (e_a \wedge e^b)) - \eta^b{}_a\left[\gamma\alpha * \mathcal{D}\alpha + d * dQ + q\alpha^{2-2q} * \left(\mathbf{\Phi}^\dagger\mathcal{D}\mathbf{\Phi} + (\mathcal{D}\mathbf{\Phi})^\dagger\mathbf{\Phi}\right)\right] = 0\,,$$

can be decomposed into its symmetrical and anti-symmetrical parts as before. The anti-symmetrical part is solved for the torsion 2-forms as

$$T^a = e^a \wedge \frac{d\alpha}{\alpha} + e^a \wedge Q\,,$$

while the symmetrical part gives the Q-field equation

$$d * dQ + \gamma\alpha * \mathcal{D}\alpha + q\alpha^{2-2q} * \left(\mathbf{\Phi}^\dagger\mathcal{D}\mathbf{\Phi} + (\mathcal{D}\mathbf{\Phi})^\dagger\mathbf{\Phi}\right) = 0\,. \tag{4.10}$$

We note that the potentials $A, \mathbf{A}$ decouple from the Q-field equation (4.10). These gravitational field equations are coupled to the Yang-Mills-Higgs equations obtained by varying $\mathbf{\Phi}, \mathbf{A}, A$:

$$\mathcal{D}_A * \mathbf{F} + \frac{g}{2}\alpha^{2-2q} * \left((\mathcal{D}\mathbf{\Phi})^\dagger\frac{\mathbf{t}_j}{2i}\mathbf{\Phi} - \mathbf{\Phi}^\dagger\frac{\mathbf{t}_j}{2i}\mathcal{D}\mathbf{\Phi}\right)\mathbf{T}_j = 0\,, \tag{4.11}$$

$$d * dA - i\frac{g'}{4}\alpha^{2-2q} * \left((\mathcal{D}\mathbf{\Phi})^\dagger\mathbf{\Phi} - \mathbf{\Phi}^\dagger\mathcal{D}\mathbf{\Phi}\right) = 0\,, \tag{4.12}$$

$$\frac{1}{2}\mathcal{D}(\alpha^{2-2q} * \mathcal{D}\mathbf{\Phi}) - \frac{\partial V}{\partial\mathbf{\Phi}^\dagger} * 1 = 0\,. \tag{4.13}$$

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By contrast Q decouples from both the gauge field equations (4.11), (4.12).

A stationary vacuum solution is $g = \eta, \mathbf{A} = 0, A = 0, Q = 0, \alpha = \alpha_0$ and

$$\mathbf{\Phi}_0 = \begin{pmatrix} 0 \\ v \end{pmatrix} {\alpha_0}^q \,, \tag{4.14}$$

so that $|\mathbf{\Phi}_0|^2 = v^2 {\alpha_0}^{2q}$. As in the previous section, to break the symmetry group an effective linearisation about this vacuum is required:

$$\begin{aligned} \alpha &= \alpha_0 + \epsilon\hat{\alpha}\,, \qquad Q = \epsilon\hat{Q}\,, \qquad g = \eta + \epsilon^2\hat{g}\,, \\ \mathbf{A} &= \epsilon\hat{\mathbf{A}}\,, \qquad A = \epsilon\hat{A}\,, \\ \mathbf{\Phi} &= \begin{pmatrix} \epsilon\hat{\phi}_+ \\ v{\alpha_0}^q - \frac{\gamma{\alpha_0}^{q-1}}{qv}\epsilon\hat{\alpha} \end{pmatrix}, \end{aligned} \tag{4.15}$$

where $\hat{\alpha}$ is real and $\hat{\phi}_+$ is complex. The linearised Q-field equation is

$$*d*d\hat{Q} + (\gamma{\alpha_0}^2 + q^2{\alpha_0}^2 v^2)\hat{Q} = 0 \tag{4.16}$$

implying $d*\hat{Q} = 0$, and the Proca mass for the $\hat{Q}$ field is given exactly as in the last section:

$${M_{\hat{Q}}}^2 = \frac{\hbar^2{\alpha_0}^2}{c^2}(\gamma + q^2v^2)\,. \tag{4.17}$$

The linearisation of the A and $\mathbf{A}$ field equations follows as in standard electro-weak theory yielding W-bosons $W^\pm = \hat{A}_1 \pm i\hat{A}_2$ with masses

$${M_W}^2 = \frac{\hbar^2v^2}{4c^2}{\alpha_0}^2 g^2\,, \tag{4.18}$$

a Z-boson $Z^0 = g\hat{A}_3 - g'\hat{A}$ with mass

$${M_Z}^2 = \frac{\hbar^2v^2}{4c^2}{\alpha_0}^2(g^2 + g'^2)\,, \tag{4.19}$$

and a massless photon $\gamma = g'\hat{A}_3 + g\hat{A}$,

$$M_\gamma = 0\,. \tag{4.20}$$

The $W^\pm$ and Z^0 fields satisfy $d*W^\pm = 0$ and $d*Z^0 = 0$. As a result of the residual U(1) local symmetry the massless photon gives rise to electric current conservation and the identification of electric charge.

It follows from above that the linearised field equations for $\hat{\phi}_+$ and $\hat{\alpha}$ are decoupled:

$$\begin{aligned} *d*d\hat{\alpha} + \left(\frac{8q^2\lambda{\alpha_0}^2v^4}{\gamma} + 8\lambda{\alpha_0}^2v^2\right)\hat{\alpha} &= 0\,, \\ d*d\hat{\phi}_+ &= \frac{vg{\alpha_0}^q}{2} * d * (\hat{A}_1 - i\hat{A}_2)\,. \end{aligned} \tag{4.21}$$

From (4.21) we note that the $\hat{\phi}_+$-excitation is not independent and can be determined in terms of the SU(2)-potentials and appropriate boundary conditions. The remaining equation for $\hat{\alpha}$ determines the scalar boson mass in terms of the parameters of this model:

$${M_{\hat{\alpha}}}^2 = \frac{\hbar^2 8\lambda{\alpha_0}^2v^2}{c^2}\left(1 + \frac{q^2v^2}{\gamma}\right). \tag{4.22}$$

In this picture the mass of the Higgs scalar $\hat{\alpha}$ depends on both the Weyl charge q of $\mathbf{\Phi}$ and the "dilaton" coupling γ, an explicit recognition of its gravitational pedigree.

5. Conclusion

It has been shown that the breakdown of local Weyl symmetry in a theory of gravity can be accommodated in the context of the standard model of particle interactions. A natural setting for this mechanism is a spacetime geometry described by a connection with dynamical torsion and a metric that is not covariantly constant. Together with a scalar field such a connection encodes new gravitational interactions that can be reformulated in terms of the standard description of einsteinian gravity. The emergence of spacetime torsion, dependent on the gradient of the dynamic scalar field, is responsible for the appearance of the so-called *improved stress-energy tensor*. It has long ago been noted [17] that this consequence of Weyl symmetry results in "improvements" to perturbative calculations involving gravitons. The effects of such improvements deserve further attention in quantised theories that involve broken Weyl symmetries to see whether unitarity can be maintained in a perturbative environment or whether such theories should be regarded as low energy effective limits of some underlying structure. On the other hand, in the broken phase in which electroweak phenomenology is discussed the theory gives rise to a Higgs particle with mass M_η and a new electrically neutral vector boson with mass M_Q such that

$$\frac{M_{\hat{\alpha}}^2}{M_Q^2} = \frac{8\lambda v^2}{\gamma}$$

in terms of the couplings in the theory. It is of interest to note that a number of grand unified models predict a new neutral vector boson and according to [20] experimental data are now detailed enough to check for its existence. It appears that such data are better described if the presence of such a boson is assumed.

The theory in this paper has been analysed in a broken phase in which normal gravitation (based on metric perturbations about Minkowski spacetime) is negligible. The mass generation mechanism has been connected with a component of non-einsteinian gravitation associated with the Weyl 1-form Q. Although the relevance of Weyl symmetry to mass generation has been noted before [21, 22, 23] we believe that the approach adopted in this paper is new. The Weyl 1-form is part of the natural spacetime geometry determined from our action principle and may be expected to give rise to new kinds of force on classical particles or in cosmological dynamics. The interaction potential V that simulates the symmetry breakdown is also dependent on a "dilatonic" scalar and this can play an intimate role in the non-perturbative aspects of the theory. As has been noted elsewhere such scalars may determine the dependence of certain "constants" of nature on the cosmological epoch or other gravitational phases. However, if the neutral boson described by the excitation of the Weyl potential Q in Minkowski spacetime can be observed in current electroweak data it may signal that a new component of gravitation can influence phenomenology at energies well above the Planck scale.

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References

[1] H. Weyl, *Gravitation and electricity*, *Preuss. Akad. Wiss. Sitzungsber.* (1918) 465.

[2] H. Weyl, *Electron and gravitation, 1. (In german)*, *Z. Phys.* **56** (1929) 330.

[3] A.S. Eddington, *The mathematical theory of relativity*, Cambridge University Press, Cambridge 1923.

[4] E. Cartan, *Sur une classe d'éspaces de Weyl*, *Ann. Sci. Ec. Norm. Sup.* **60** (1943) 1.

[5] I.M. Benn, R.W. Tucker, *An introduction to spinors and geometry with applications in physics*, Adam Hilger, 1987.

[6] E.J. Post, *Formal structure of electromagnetics*, 2nd edition, Dover, 1997.

[7] P. Szekeres, *The mathematical foundations of dimensional analysis and the question of fundamental units*, *Int. J. Theor. Phys.* **17** (1978) 951.

[8] P.A.M. Dirac, *New basis for cosmology*, *Proc. Roy. Soc.* **A165** (1938) 199.

[9] P.A.M. Dirac, *Long range forces and broken symmetries*, *Proc. Roy. Soc.* **A333** (1973) 403.

[10] P.A.M. Dirac, *The large numbers hypothesis and the Einstein theory of gravitation*, *Proc. Roy. Soc.* **A365** (1979) 19.

[11] L.L. Smalley, *Brans-Dicke type models with nonmetricity*, *Phys. Rev.* **D 33** (1986) 3590.

[12] K.P. Tod, *Compact 3-dimensional Einstein-Weyl structures*, *J. London Math. Soc.* **45** (1992) 341.

[13] K.P. Tod, *Local heterotic geometry and selfdual Einstein-Weyl spaces*, *Class. and Quant. Grav.* **13** (1996) 2609.

[14] C. Brans and R.H. Dicke, *Mach's principle and a relativistic theory of gravitation*, *Phys. Rev.* **124** (1961) 925.

[15] R.H. Dicke, *Mach's principle and invariance under transformation of units*, *Phys. Rev.* **125** (1961) 2163.

[16] R.H. Dicke, *Experimental relativity*, in *Relativity, groups and topology. 1963 Les Houches Lectures*, B.S. De Witt ed., North-Holland, 1964.

[17] S.R. Coleman and R. Jackiw, *Why dilatation generators do not generate dilatations?*, *Ann. Phys. (NY)* **67** (1971) 552.

[18] T. Dereli and R.W. Tucker, *Weyl scalings and spinor matter interactions in scalar-tensor theories of gravitation*, *Phys. Lett.* **B 110** (1982) 206.

[19] J.L. Anderson, *Scale invariance of the second kind and the Brans-Dicke scalar-tensor theory*, *Phys. Rev.* **D 3** (1971) 1689.

[20] J. Erler and P. Langacker, *Indications for an extra neutral gauge boson in electroweak precision data*, *Phys. Rev. Lett.* **84** (2000) 212 [hep-ph/9910315].

[21] F. Mansouri, *Conformal gravity as a gauge theory*, *Phys. Rev. Lett.* **42** (1979) 1021.

[22] W. Drechsler and H. Tann, *Broken Weyl invariance and the origin of mass*, *Found. Phys.* **29** (1999) 1023 [gr-qc/9802044].

[23] W. Drechsler, *Mass generation by Weyl symmetry breaking*, *Found. Phys.* **29** (1999) 1327 [gr-qc/9901030].

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